2nd Exam Calculus of Variations & Optimal Control 2015-16

Datum	:	08-04-2016
Plaats	:	BB293
Tijd	:	9.00-12.00

You need to motivate all your answers and to mention clearly which theorems etc. you are using.

The exam is OPEN BOOK.

1. Consider the scalar system

$$\dot{x} = -ax + u, \quad x(0) = x_0, \quad |u(t)| \leq M$$

for a constant a and positive constant M, and cost criterion

$$J(u) = \frac{1}{2}\int_0^1 u^2(t)dt + x(1)$$

(a) Determine the Hamiltonian for this optimal control problem, and the differential equation for the co-state p.

(b) What is the optimal input (as function of time) for the case $M = \infty$?

(c) What is the optimal input for finite M ? Distinguish between the case $a \geq 0$ and $a < 0$.

2. Consider the differential equation describing a mathematical pendulum

$$mR^2\ddot{\phi}(t) + gmR\sin(\phi(t)) = u(t),$$

where ϕ is the angle with respect to the stable equilibrium state, $u(\cdot)$ is a torque exerted around the suspension point. As output we take the angular velocity; $y(t) =$.

The objective is to minimize the cost

$$J_{[0,t_e]}(x_0, u) = mR^2 y(t_e)^2 - 2mgR\cos(\phi(t_e)) + \int_0^{t_e} y^2(t) + u^2(t)dt.$$

We introduce the notation $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \phi \\ \dot{\phi} \end{pmatrix}$.

a. Determine the state differential equation.

b. Determine the Hamiltonian $H(p, x, u)$ and the differential equation for the co-state.

c. Calculate $\int_0^{t_e} y(t)u(t)dt$. What do you see?

d. Give an expression for the value function.

e. Use Theorem 57 to find an expression for the co-state. Verify that it satisfies the differential equation you already found.

3. Consider a mass-spring system (q the elongation of the spring, $p = m\dot{q}$ the momentum of the mass)

$$\begin{aligned} \dot{q} &= \tfrac{1}{m}p \\ \dot{p} &= -kq + u \end{aligned}$$

with m the mass, $k > 0$ the spring constant, and u the external force on the mass.

   (a) Prove that the feedback $u = -\dot{q}$ renders the equilibrium $(q, p) = (0, 0)$ asymptotically stable.

   (b) In practice the magnitude of the input u is often constrained, say $|u| \leq M$ for some constant $M > 0$. Define the function sgn by $\text{sgn}(z) = 1$ if $z > 0$, $\text{sgn}(z) = -1$ if $z < 0$, and $\text{sgn}(0) = 0$.

   Show that in this case the equilibrium is still rendered asymptotically stable by the feedback

$$\begin{aligned} u &= -\dot{q}, && \text{indien } |\dot{q}| \leq M \\ u &= -M\text{sgn}(\dot{q}), && \text{indien } |\dot{q}| > M \end{aligned}$$

   using the Lyapunov function (= total energy) $V(q, p) = \frac{1}{2m}p^2 + \frac{1}{2}kq^2$.

   (c) Is $(0, 0)$ a *globally* asymptotically stable equilibrium of the closed-loop system resulting from applying the feedback in (b) ?

4. Consider the problem of extremizing

$$\int_0^T [\frac{1}{2}m\dot{q}^2(t) - V(q(t))]dt$$

over all functions $q : [0, T] \to \mathbb{R}^n$ with $q(0) = q_0, q(T) = q_T$ fixed. Here $V : \mathbb{R}^n \to \mathbb{R}$ is an arbitrary differentiable function. Derive the Euler-Lagrange equation. (This corresponds to the motion of a particle with mass m in $\mathbb{R}^n$ subject to a potential field with potential energy V.)

What is the Beltrami identity for this case ? Give an interpretation.

Distribution of points: Total 100; Free 10

1. a: 5, b: 10, c: 10.

2. a: 5, b: 5, c: 5, d: 10, e: 5.

3. a: 5, b: 10, c: 10.

4. 10